The Gauss Map of Hypersurfaces in 2-Step Nilpotent Lie Groups

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In this paper we consider smooth oriented hypersurfaces in 2-step nilpotent Lie groups with a left invariant metric and derive an expression for the Laplacian of the Gauss map for such hypersurfaces in the general case and in some particular cases. In the case of CMC-hypersurface in the $2m+1$-dimensional Heisenberg group we also derive necessary and sufficient conditions for the Gauss map to be harmonic and prove that for $m=1$ all CMC-surfaces with the harmonic Gauss map are "cylinders".
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It is proved in [12] that the Gauss map of a smooth n-dimensional oriented hypersurface in $\mathbb{R}^{n+1}$ is harmonic if and only if the hypersurface is of a constant mean curvature (CMC). The same is proved for the cases of S^3, which is a Lie group and thus has a natural definition of the Gauss map [9], and, in different settings, of H^3 [10]. A generalization of this proposition to the case of Lie groups with a bi-invariant metric (this class of Lie groups includes, for example, abelian groups $\mathbb{R}^{n+1}$ and $S^3 \cong SU(2)$) is proved in [6]. In this paper we use methods of [6] for an investigation of the Gauss map of a hypersurface in some 2-step nilpotent Lie group with a left invariant metric. The theory of such groups is highly developed (see, for example, [3] and [4]).

The paper is organized as follows. After some preliminary information (section 1), in section 2 we obtain an expression for the Laplacian of the Gauss map of a hypersurface in a 2-step nilpotent Lie group (Theorem 1).

Using this expression we prove some facts concerning relations between harmonic properties of the Gauss map and the mean curvature of the hypersurface (see section 3), in particular, a sufficient condition for the stability of CMC-hypersurfaces (Proposition 6). In section 4 we consider the cases of Heisenberg type groups and Heisenberg groups. We show the harmonicity of the Gauss map of a hypersurface in such groups is, in general, not equivalent to the constancy of the mean curvature. Also we obtain necessary and sufficient conditions for this equivalence in the particular case of Heisenberg groups (Proposition 7).

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1 Preliminaries

Let us recall some basic definitions and facts about the stability of constant mean curvature hypersurfaces in Riemannian manifolds. Suppose M is a smooth n-dimensional manifold immersed in a smooth $n+1$-dimensional Riemannian manifold as a CMC-hypersurface. Denote by η a unit normal vector field of M. Let $D \subset M$ be a compact domain. The *index form* of D is a quadratic form $Q(\cdot,\cdot)$ on $C^\infty(D)$ defined by the equation

$$Q(w,w) = -\int\limits_D wLw\, dV_M, \tag{1}$$

where dV_M is the volume form of the induced metric on M, L is the *Jacobi operator* $\Delta_M + \Big(Ric(\eta,\eta) + \|B\|^2\Big)$, $\mathrm{Ric}(\cdot,\cdot)$ is the Ricci tensor of the ambient manifold, $\|B\|$ is the norm of the second fundamental form of the immersion, and Δ_M is the Laplacian of the induced metric (see, for example, [2]).

Let M be a minimal hypersurface (a hypersurface of a nonzero constant mean curvature, respectively). A compact domain $D \subset M$ is called *stable* if $Q(w,w) \geqslant 0$ for every function $w \in C^\infty(D)$ vanishing on ∂D (for every $w \in C^\infty(D)$ vanishing on ∂D and with $\int\limits_D w\, dV_M = 0$, respectively). The hypersurface M is *stable* if every compact domain $D \subset M$ is stable, and is *unstable* otherwise (see, for example, [1]). It is proved in [7, Theorem 1] that if the *Jacobi equation* $Lw = 0$ admits a solution w strictly positive on M, then M is stable.

Let (M, g) be a smooth Riemannian manifold. Denote by Δ_M the Laplacian of g. For each $\phi \in C^\infty(M, S^n)$ denote by $\Delta_M \phi$ the vector $(\Delta_M \phi_1, \ldots, \Delta_M \phi_{n+1})$, where $(\phi_1, \ldots, \phi_{n+1})$ is the coordinate functions of ϕ for the standard embedding of a unit sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$. It is well known that the harmonicity of ϕ is equivalent to the equation $\Delta_M \phi = 2e(\phi)\phi$, where $e(\phi)$ is the energy density function of ϕ (see [14, p. 140, Corollary (2.24)]).

Suppose M is an oriented hypersurface in a $n+1$-dimensional Lie group N with a left invariant Riemannian metric. Fix the unit normal vector field η of M with respect to the orientation. Let p be a point of M. Denote by L_a the left translation by $a \in N$, and let dL_a be the differential of this map. We can consider p as an element of N if we identify this point with its image under the immersion. Let G be the map of M to $S^n \subset \mathcal{N}$ such that $G(p) = (dL_p)^{-1}(\eta(p))$ for all $p \in N$, where $\mathcal{N}$ is the Lie algebra of N. We call G the *Gauss map* of M. It is proved in [6] that if a metric of N is bi-invariant (see [11] on a structure of such Lie groups), then the Gauss map is harmonic if and only if the mean curvature of M is constant.

Now we consider the case of nilpotent Lie groups. Let $\mathcal{N}$ be a finite dimensional Lie algebra over $\mathbb{R}$ with a Lie bracket $[\cdot, \cdot]$. The lower central series of $\mathcal{N}$ is defined inductively by $\mathcal{N}^1 = \mathcal{N}$, $\mathcal{N}^{k+1} = \left[\mathcal{N}^k, \mathcal{N}\right]$ for all positive integers k. The Lie algebra $\mathcal{N}$ is called *k-step nilpotent* if $\mathcal{N}^k \neq 0$ and $\mathcal{N}^{k+1} = 0$. A Lie group N is called *k-step nilpotent* if its Lie algebra $\mathcal{N}$ is k-step nilpotent.

In the sequel, we consider a 2-step nilpotent connected and simply connected Lie group N and its Lie algebra $\mathcal{N}$. Let $\mathcal{Z}$ be the center of $\mathcal{N}$. Since $\mathcal{N}$ is 2-step nilpotent, $0 \neq [\mathcal{N}, \mathcal{N}] \subset \mathcal{Z}$. Suppose that $\mathcal{N}$ is endowed with a scalar product $\langle \cdot, \cdot \rangle$. This scalar product induces a left invariant Riemannian metric on N, which we also denote by $\langle \cdot, \cdot \rangle$. Let $\mathcal{V}$ be an orthogonal complement to $\mathcal{Z}$ in $\mathcal{N}$ with respect to $\langle \cdot, \cdot \rangle$. Then $[\mathcal{V}, \mathcal{V}] = [\mathcal{N}, \mathcal{N}] \subset \mathcal{Z}$. For each $Z \in \mathcal{Z}$ a linear operator $J(Z) \colon \mathcal{V} \to \mathcal{V}$ is well defined by $\langle J(Z)X, Y \rangle = \langle [X, Y], Z \rangle$, where $X, Y \in \mathcal{V}$ are arbitrary vectors.

An important class of 2-step nilpotent groups consists of so-called $2m+1$-dimensional *Heisenberg groups*, which appear in some problems of quantum and Hamiltonian mechanics [8]. The Lie algebra of a Heisenberg group has a basis $K_1, \ldots, K_m$, $L_1, \ldots, L_m$, Z and the structure relations

$$[K_i, L_j] = \delta_{ij} Z,\ [K_i, K_j] = [L_i, L_j] = [K_i, Z] = [L_i, Z] = 0,\ 1 \leqslant i, j \leqslant m,$$

where δ_{ij} is the Kronecker symbol. We introduce a scalar product such that this basis is orthonormal. The three-dimensional Heisenberg group with a

left invariant Riemannian metric is often denoted by Nil and is a three-dimensional Thurston geometry. A Lie algebra $\mathcal{N}$ is of *Heisenberg type* if $J(Z)^2 = -\langle Z, Z\rangle \operatorname{Id}|_{\mathcal{V}}$, for every $Z \in \mathcal{Z}$ [4]. Its Lie group N is called a Lie group of *Heisenberg type.* This class of groups contains, for example, Heisenberg groups and quaternionic Heisenberg groups [3, p. 617]. A general approach to the structure of 2-step nilpotent Lie algebras was developed in the paper [5].

The Riemannian connection associated with $\langle\cdot,\cdot\rangle$ is defined on left invariant fields by (see [3])

$$\begin{array}{lcll}
\nabla_X Y & = & \frac{1}{2}[X,Y], & X, Y \in \mathcal{V}; \\
\nabla_X Z = \nabla_Z X & = & -\frac{1}{2}J(Z)X, & X \in \mathcal{V},\ Z \in \mathcal{Z}; \\
\nabla_Z Z^* & = & 0, & Z, Z^* \in \mathcal{Z}.
\end{array} \tag{2}$$

From this one can obtain for the curvature tensor

$$\begin{array}{lcll}
R(X,Y)X^* & = & \frac{1}{2}J([X,Y])X^* & \\
 & & -\frac{1}{4}J([Y,X^*])X & X, X^*, Y \in \mathcal{V}; \\
 & & +\frac{1}{4}J([X,X^*])Y, & \\
R(X,Z)Y & = & -\frac{1}{4}[X, J(Z)Y], & \\
R(X,Y)Z & = & -\frac{1}{4}[X, J(Z)Y] & X, Y \in \mathcal{V},\ Z \in \mathcal{Z}; \\
 & & +\frac{1}{4}[Y, J(Z)X], & \\
R(X,Z)Z^* & = & -\frac{1}{4}J(Z)J(Z^*)X, & \\
R(Z,Z^*)X & = & -\frac{1}{4}J(Z^*)J(Z)X & X \in \mathcal{V},\ Z, Z^* \in \mathcal{Z}; \\
 & & +\frac{1}{4}J(Z)J(Z^*)X, & \\
R(Z,Z^*)Z^{**} & = & 0, & Z, Z^*, Z^{**} \in \mathcal{Z}.
\end{array} \tag{3}$$

And the Ricci tensor is defined by

$$\begin{array}{lcll}
\operatorname{Ric}(X,Y) & = & \frac{1}{2}\sum_{k=1}^{l}\langle J(Z_k)^2 X, Y\rangle, & X, Y \in \mathcal{V}; \\
\operatorname{Ric}(X,Z) & = & 0, & X \in \mathcal{V},\ Z \in \mathcal{Z}; \\
\operatorname{Ric}(Z,Z^*) & = & -\frac{1}{4}\operatorname{Tr}(J(Z)J(Z^*)), & Z, Z^* \in \mathcal{Z}.
\end{array} \tag{4}$$

Here $\dim \mathcal{Z} = l$, and $Z_1, \dots, Z_l$ is an orthonormal basis for $\mathcal{Z}$.

2 The Laplacian of the Gauss map

Suppose $\dim N = \dim \mathcal{N} = n + 1$, $\dim \mathcal{Z} = n - q + 1$, where n and q are positive integers, $q \leqslant n$.

Let M be a smooth oriented manifold, $\dim M = n$. Suppose $M \to N$ is an immersion of this manifold in N as a hypersurface, and η is the unit normal vector field of M in N. For each point p of M, suppose that $\eta(p) = Y_{n+1} = X_{n+1} + Z_{n+1}$, where $X_{n+1} \in \mathcal{V}$, $Z_{n+1} \in \mathcal{Z}$. Throughout this paper, we denote by X_i, Y_i, Z_i elements of T_pN as well as the corresponding left invariant vector fields, which are elements of $\mathcal{N}$. Choose an orthonormal frame $\{Y_1, \dots, Y_n\}$ in the vector space $T_pM \subset T_pN$ such that for $1 \leqslant i \leqslant q-1$ $Y_i = X_i$, $Y_q = X_q - Z_q$, and for $q+1 \leqslant i \leqslant n$ $Y_i = Z_i$, where $X_1, \dots, X_q$ are elements of $\mathcal{V}$, $Z_q, \dots, Z_n$ belong to $\mathcal{Z}$, $X_{n+1} = \lambda X_q$, $Z_{n+1} = \mu Z_q$, where $\lambda \geqslant 0$ and $\mu \geqslant 0$, $|X_q| = |Z_{n+1}|$, $|Z_q| = |X_{n+1}|$. Let $E_1, \dots E_n$ be an orthonormal frame defined on some neighborhood U of p such that $E_i(p) = Y_i$ and $\left(\nabla_{E_i} E_j\right)^T(p) = 0$, for all $i, j = 1, \dots n$ (such a frame is called geodesic at p). Here we denote by $(\cdot)^T$ the projection to T_pM.

We can rewrite (4) in the following form

$$
\begin{array}{rcll}
\operatorname{Ric}(X,Y) & = & \frac{1}{2}\sum\limits_{k=q}^{n+1}\langle J(Z_k)^2 X, Y\rangle, & X, Y \in \mathcal{V}; \\
\operatorname{Ric}(X,Z) & = & 0, & X \in \mathcal{V},\ Z \in \mathcal{Z}; \\
\operatorname{Ric}(Z,Z^*) & = & -\frac{1}{4}\sum\limits_{1\leqslant k\leqslant q,\, k=n+1}\langle J(Z)J(Z^*)X_k, X_k\rangle, & Z, Z^* \in \mathcal{Z}.
\end{array} \tag{5}
$$

In particular, for all $X, Y \in \mathcal{V}$

$$
\begin{aligned}
&\sum_{1\leqslant i\leqslant q,\, i=n+1}\langle J([X,X_i])X_i, Y\rangle \\
&= \sum_{1\leqslant i\leqslant q,\, i=n+1}\sum_{j=q}^{n+1}\langle [X,X_i], Z_j\rangle\langle [X_i,Y], Z_j\rangle \\
&= -\sum_{j=q}^{n+1}\sum_{1\leqslant i\leqslant q,\, i=n+1}\langle J(Z_j)X, X_i\rangle\langle J(Z_j)Y, X_i\rangle \\
&= \sum_{j=q}^{n+1}\langle J(Z_j)^2 X, Y\rangle = 2\operatorname{Ric}(X,Y).
\end{aligned} \tag{6}
$$

For $1 \leqslant i, j \leqslant n$, denote by $b_{ij} = \langle \nabla_{E_i} E_j, \eta\rangle$ the coefficients of the second fundamental form of the immersion, by $\|B\|$ the norm of this form, and by H the mean curvature of the immersion on U. Since the frame is orthonormal over U,

$$
\begin{aligned}
H &= \frac{1}{n}\sum_{i=1}^{n} b_{ii}, \\
\|B\|^2 &= \sum_{1\leqslant i,j\leqslant n}(b_{ij})^2.
\end{aligned} \tag{7}
$$

Suppose that on U

$$\eta = \sum_{j=1}^{n+1} a_j Y_j,$$

where $\{a_j\}_{j=1}^{n+1}$ are some functions on U. It is clear that $a_j(p) = \delta_{j\,n+1}$. Then the Gauss map $G\colon U \to S^n \subset \mathbb{R}^{n+1}$ takes the form

$$G = \sum_{j=1}^{n+1} a_j Y_j(e).$$

In particular, $G(p) = Y_{n+1}(e)$. Denote by Δ the Laplacian Δ_M of the induced metric on M.

Theorem 1. *Let M be a smooth oriented manifold immersed in a 2-step nilpotent Lie group N as a hypersurface and G be the Gauss map of M. Then, in the above notation*

$$\begin{aligned}
\Delta G(p) = \sum_{k=1}^{q} \Bigg(& -Y_k(nH) + \sum_{j=1}^{q-1} \langle J([X_k, X_j])X_j, X_{n+1}\rangle \\
& + 4\langle R(X_k, Z_{n+1})Z_{n+1}, X_{n+1}\rangle - 2\sum_{i=1}^{q}\sum_{j=q+1}^{n} b_{ij}(p)\langle J(Z_j)X_i, X_k\rangle \\
& + 2\sum_{i=1}^{q} b_{iq}(p)\langle J(Z_q)X_i, X_k\rangle + nH(p)\langle J(Z_{n+1})X_{n+1}, X_k\rangle \Bigg) Y_k(e) \\
& + \sum_{k=q+1}^{n} \Big(-Y_k(nH) \Big) Y_k(e) \\
+ \Bigg(& \sum_{j=1}^{q-1} \langle J([X_{n+1}, X_j])X_j, X_{n+1}\rangle + 4\langle R(X_{n+1}, Z_{n+1})Z_{n+1}, X_{n+1}\rangle \\
& - 2\sum_{i=1}^{q}\sum_{j=q+1}^{n} b_{ij}(p)\langle J(Z_j)X_i, X_{n+1}\rangle + 2\sum_{i=1}^{q} b_{iq}(p)\langle J(Z_q)X_i, X_{n+1}\rangle \\
& - \|B\|^2(p) - \mathrm{Ric}(Y_{n+1}, Y_{n+1}) \Bigg) Y_{n+1}(e).
\end{aligned} \tag{8}$$

Here $Y_k(nH)$ denotes the derivative of the function nH with respect to the vector field Y_k.

P r o o f. Since the frame $E_1, \ldots, E_n$ is geodesic at p, the Laplacian at this point has the form

$$\Delta G(p) = \sum_{j=1}^{n+1} \sum_{i=1}^{n} E_i E_i(a_j) Y_j(e). \tag{9}$$

For $1 \leqslant i \leqslant n$ we have on U

$$\nabla_{E_i} \eta = \sum_{j=1}^{n+1} E_i(a_j) Y_j + \sum_{j=1}^{n+1} a_j \nabla_{E_i} Y_j, \tag{10}$$

$$\nabla_{E_i} \nabla_{E_i} \eta = \sum_{j=1}^{n+1} E_i E_i(a_j) Y_j + 2 \sum_{j=1}^{n+1} E_i(a_j) \nabla_{E_i} Y_j + \sum_{j=1}^{n+1} a_j \nabla_{E_i} \nabla_{E_i} Y_j. \tag{11}$$

Considering this expression at p and taking its scalar product with Y_k for $1 \leqslant k \leqslant n+1$, we get

$$\langle \nabla_{E_i} \nabla_{E_i} \eta, Y_k \rangle = E_i E_i(a_k) + 2 \sum_{j=1}^{n+1} E_i(a_j) \langle \nabla_{E_i} Y_j, Y_k \rangle + \langle \nabla_{E_i} \nabla_{E_i} Y_{n+1}, Y_k \rangle.$$

Then for scalar coefficients in (9) we have

$$\begin{gathered} \sum_{i=1}^{n} E_i E_i(a_k) = \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla_{E_i} \eta, Y_k \rangle \\ -2 \sum_{j=1}^{n+1} \sum_{i=1}^{n} E_i(a_j) \langle \nabla_{E_i} Y_j, Y_k \rangle - \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla_{E_i} Y_{n+1}, Y_k \rangle. \end{gathered} \tag{12}$$

For $1 \leqslant k \leqslant n$, the first expression in (7) and the definition of a second fundamental form imply at p

$$Y_k(nH) = E_k \left(\sum_{i=1}^{n} \langle \nabla_{E_i} E_i, \eta \rangle \right) = \sum_{i=1}^{n} \langle \nabla_{E_k} \nabla_{E_i} E_i, \eta \rangle$$

$$= \sum_{i=1}^{n} \left(\langle R(E_k, E_i) E_i, \eta \rangle + \langle \nabla_{E_i} \nabla_{E_k} E_i, \eta \rangle + \langle \nabla_{[E_k, E_i]} E_i, \eta \rangle \right)$$

$$= \sum_{i=1}^{n} \left(\langle R(Y_k, Y_i) Y_i, Y_{n+1} \rangle + \langle \nabla_{E_i} \nabla_{E_k} E_i, \eta \rangle \right).$$

The second equality in the equation above follows from the fact that the projection $(\nabla_{E_i} E_i)^T = 0$ at p and the vector $\nabla_{E_k}\eta$ is tangent to M. The fourth equality is a consequence of

$$[E_k, E_i] = ([E_k, E_i])^T = (\nabla_{E_k} E_i - \nabla_{E_i} E_k)^T = 0$$

at p. Since $\langle [E_k, E_i], \eta\rangle = 0$ on U, and $[E_k, E_i](p) = 0$, at p we have

$$0 = \langle \nabla_{E_i}[E_k, E_i], \eta\rangle = \langle \nabla_{E_i}\nabla_{E_k} E_i - \nabla_{E_k}\nabla_{E_i} E_i, \eta\rangle,$$

for $1 \leqslant i \leqslant n$, hence

$$Y_k(nH) = \sum_{i=1}^{n} \left(\langle R(Y_k, Y_i)Y_i, Y_{n+1}\rangle + \langle \nabla_{E_i}\nabla_{E_i} E_k, \eta\rangle \right). \tag{13}$$

Differentiating $\langle E_k, \eta\rangle = 0$ two times with respect to E_i (here we put $1 \leqslant k, i \leqslant n$) and using $\langle \nabla_{E_i} E_k, \nabla_{E_i}\eta\rangle(p) = 0$, we derive from (13)

$$\begin{aligned} \sum_{i=1}^{n} \langle \nabla_{E_i}\nabla_{E_i}\eta, Y_k\rangle &= -\sum_{i=1}^{n} \langle \nabla_{E_i}\nabla_{E_i} E_k, \eta\rangle \\ = -Y_k(nH) + \sum_{i=1}^{n} \langle R(Y_k, Y_i)Y_i, Y_{n+1}\rangle &= -Y_k(nH) + \mathrm{Ric}(Y_k, Y_{n+1}). \end{aligned} \tag{14}$$

For $1 \leqslant i \leqslant n$, differentiating $\langle \eta, \eta\rangle = 1$ two times with respect to E_i, we get $2\langle \nabla_{E_i}\nabla_{E_i}\eta, \eta\rangle + 2\langle \nabla_{E_i}\eta, \nabla_{E_i}\eta\rangle = 0$. This equation and the second expression in (7) imply at p

$$\begin{aligned} \sum_{i=1}^{n} \langle \nabla_{E_i}\nabla_{E_i}\eta, Y_{n+1}\rangle &= -\sum_{i=1}^{n} \langle \nabla_{E_i}\eta, \nabla_{E_i}\eta\rangle \\ &= -\sum_{i=1}^{n}\sum_{j=1}^{n} \langle \nabla_{E_i}\eta, E_j\rangle \langle \nabla_{E_i}\eta, E_j\rangle \\ &= -\sum_{1\leqslant i,j\leqslant n} \langle \nabla_{E_i} E_j, \eta\rangle^2 = -\|B\|^2(p). \end{aligned} \tag{15}$$

Consider the scalar products $\langle \nabla_{E_i}\eta, Y_k\rangle$ at the point p, for $1 \leqslant i \leqslant n$, $1 \leqslant k \leqslant n+1$. As $|\eta| = |Y_{n+1}| = 1$, we obtain from (10)

$$0 = \langle \nabla_{E_i}\eta, \eta\rangle(p) = \langle \nabla_{E_i}\eta, Y_{n+1}\rangle = E_i(a_{n+1}) + \langle \nabla_{E_i} Y_{n+1}, Y_{n+1}\rangle = E_i(a_{n+1}).$$

For $1 \leqslant k \leqslant n$, $\langle E_k, \eta\rangle = 0$ imply

$$b_{ik}(p) = \langle \nabla_{E_i} E_k, \eta \rangle(p) = -\langle \nabla_{E_i} \eta, E_k \rangle(p) = -\langle \nabla_{E_i} \eta, Y_k \rangle$$
$$= -E_i(a_k) - \langle \nabla_{E_i} Y_{n+1}, Y_k \rangle.$$

Hence at p we have

$$-2 \sum_{j=1}^{n+1} \sum_{i=1}^{n} E_i(a_j) \langle \nabla_{E_i} Y_j, Y_k \rangle$$

$$= 2 \sum_{j=1}^{n} \sum_{i=1}^{n} \left(b_{ij}(p) + \langle \nabla_{Y_i} Y_{n+1}, Y_j \rangle \right) \langle \nabla_{Y_i} Y_j, Y_k \rangle.$$

It follows from (2) that for $1 \leqslant i, j \leqslant q$ the expression $b_{ij}(p) \langle \nabla_{X_i} X_j, Y_k \rangle$ is skew-symmetric with respect to i, j; hence the sum of such terms with respect to i and j vanishes. Sum up other expressions using the symmetry $\nabla_X Z = \nabla_Z X$ for all $X \in \mathcal{V}$, $Z \in \mathcal{Z}$ and the symmetry of the second fundamental form. We obtain

$$\begin{aligned} -2 \sum_{j=1}^{n+1} \sum_{i=1}^{n} E_i(a_j) \langle \nabla_{E_i} Y_j, Y_k \rangle &= -2 \sum_{i=1}^{q} \sum_{j=q+1}^{n} b_{ij}(p) \langle J(Z_j) X_i, Y_k \rangle \\ +2 \sum_{i=1}^{q} b_{iq}(p) \langle J(Z_q) X_i, Y_k \rangle &+ 2 \sum_{1 \leqslant i,j \leqslant n} \langle \nabla_{Y_i} Y_{n+1}, Y_j \rangle \langle \nabla_{Y_i} Y_j, Y_k \rangle. \end{aligned} \tag{16}$$

Now we can complete the proof of the theorem using the following technical lemmas.

Lemma 2. *The last summand on the right hand side of* (16) *is equal to*

$$2 \sum_{1 \leqslant i,j \leqslant n} \langle \nabla_{Y_i} Y_{n+1}, Y_j \rangle \langle \nabla_{Y_i} Y_j, Y_k \rangle$$
$$= \left[\begin{array}{ll} 2 \operatorname{Ric}(Y_k, Y_{n+1}) + 4 \langle R(X_k, Z_{n+1}) Z_{n+1}, X_{n+1} \rangle, & 1 \leqslant k \leqslant q-1; \\ 2 \operatorname{Ric}(X_q, X_{n+1}) + 2 \operatorname{Ric}(Z_q, Z_{n+1}) & \\ +4 \langle R(X_q, Z_{n+1}) Z_{n+1}, X_{n+1} \rangle & k = q; \\ -4 \langle R(X_{n+1}, Z_q) Z_{n+1}, X_{n+1} \rangle, & \\ -2 \operatorname{Ric}(Y_k, Y_{n+1}) & \\ +4 \langle R(X_{n+1}, Z_k) Z_{n+1}, X_{n+1} \rangle, & q+1 \leqslant k \leqslant n; \\ 2 \operatorname{Ric}(X_{n+1}, X_{n+1}) - 2 \operatorname{Ric}(Z_{n+1}, Z_{n+1}) & \\ +8 \langle R(X_{n+1}, Z_{n+1}) Z_{n+1}, X_{n+1} \rangle, & k = n+1. \end{array} \right. \tag{17}$$

Lemma 3. *The last summand on the right hand side of* (12) *can be reduced to the form*

$$
-\sum_{i=1}^{n}\langle\nabla_{E_i}\nabla_{E_i}Y_{n+1},Y_k\rangle
= \left[\begin{array}{ll}
nH(p)\langle J(Z_{n+1})X_{n+1},X_k\rangle-\operatorname{Ric}(Y_k,Y_{n+1}), & 1\leqslant k\leqslant q-1;\\
-\operatorname{Ric}(X_q,X_{n+1})-\operatorname{Ric}(Z_q,Z_{n+1}) & \\
+4\langle R(X_{n+1},Z_q)Z_{n+1},X_{n+1}\rangle, & k=q;\\
\operatorname{Ric}(Y_k,Y_{n+1})-4\langle R(X_{n+1},Z_k)Z_{n+1},X_{n+1}\rangle, & q+1\leqslant k\leqslant n;\\
-\operatorname{Ric}(X_{n+1},X_{n+1})+\operatorname{Ric}(Z_{n+1},Z_{n+1}) & \\
-4\langle R(X_{n+1},Z_{n+1})Z_{n+1},X_{n+1}\rangle, & k=n+1.
\end{array}\right. \tag{18}
$$

Now, if we combine (12) with (16), (17), (18), and (6), we get (8). $\square$

P r o o f o f L e m m a 2. For $1\leqslant k\leqslant q-1$ from the expressions for the Riemannian connection we get

$$
\sum_{i=1}^{n}\sum_{j=1}^{n}\langle\nabla_{Y_i}Y_{n+1},Y_j\rangle\langle\nabla_{Y_i}Y_j,X_k\rangle=\sum_{i=1}^{q-1}\Bigg(\langle\frac{1}{2}[X_i,X_{n+1}],-Z_q\rangle
$$

$$
+\langle-\frac{1}{2}J(Z_{n+1})X_i,X_q\rangle\Bigg)\langle\frac{1}{2}J(Z_q)X_i,X_k\rangle
$$

$$
+\sum_{i=1}^{q-1}\sum_{j=q+1}^{n}\langle\frac{1}{2}[X_i,X_{n+1}],Z_j\rangle\langle-\frac{1}{2}J(Z_j)X_i,X_k\rangle
$$

$$
+\sum_{j=1}^{q-1}\langle-\frac{1}{2}J(Z_{n+1})X_q+\frac{1}{2}J(Z_q)X_{n+1},X_j\rangle\langle\frac{1}{2}J(Z_q)X_j,X_k\rangle
$$

$$
+\Bigg(\langle\frac{1}{2}J(Z_q)X_{n+1}-\frac{1}{2}J(Z_{n+1})X_q,X_q\rangle+\langle\frac{1}{2}[X_q,X_{n+1}],-Z_q\rangle\Bigg)\langle J(Z_q)X_q,X_k\rangle
$$

$$
+\sum_{j=q+1}^{n}\langle\frac{1}{2}[X_q,X_{n+1}],Z_j\rangle\langle-\frac{1}{2}J(Z_j)X_q,X_k\rangle
$$

$$
+\sum_{i=q+1}^{n}\sum_{j=1}^{q-1}\langle-\frac{1}{2}J(Z_i)X_{n+1},X_j\rangle\langle-\frac{1}{2}J(Z_i)X_j,X_k\rangle
$$

$$+\sum_{i=q+1}^{n}\langle -\frac{1}{2}J(Z_i)X_{n+1},X_q\rangle\langle -\frac{1}{2}J(Z_i)X_q,X_k\rangle.$$

The skew-symmetry of J implies $\langle J(Z)X_q,X_{n+1}\rangle=-\langle J(Z)X_{n+1},X_q\rangle=0$ for all $Z\in\mathcal{Z}$, and $[X_q,X_{n+1}]=0$. Hence we can rewrite the above expression in the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\langle\nabla_{Y_i}Y_{n+1},Y_j\rangle\langle\nabla_{Y_i}Y_j,X_k\rangle$$

$$=\frac{1}{2}\sum_{j=q}^{n}\sum_{1\leqslant i\leqslant q,\, i=n+1}\langle -J(Z_j)X_{n+1},X_i\rangle\langle J(Z_j)X_k,X_i\rangle$$

$$=-\frac{1}{2}\sum_{j=q}^{n}\langle J(Z_j)X_{n+1},J(Z_j)X_k\rangle=\frac{1}{2}\sum_{j=q}^{n}\langle J(Z_j)^2X_{n+1},X_k\rangle$$

$$=\mathrm{Ric}(X_k,X_{n+1})+2\langle R(X_k,Z_{n+1})Z_{n+1},X_{n+1}\rangle.$$

This implies the first equality in (17).

For $q+1\leqslant k\leqslant n$ we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\langle\nabla_{Y_i}Y_{n+1},Y_j\rangle\langle\nabla_{Y_i}Y_j,Z_k\rangle=\sum_{i=1}^{q-1}\sum_{j=1}^{q-1}\langle -\frac{1}{2}J(Z_{n+1})X_i,X_j\rangle\langle\frac{1}{2}[X_i,X_j],Z_k\rangle$$

$$+\frac{1}{2}\sum_{i=1}^{q-1}\left(\langle -\frac{1}{2}J(Z_{n+1})X_i,X_q\rangle+\langle\frac{1}{2}[X_i,X_{n+1}],-Z_q\rangle\right)\langle\frac{1}{2}[X_i,X_q],Z_k\rangle$$

$$+\sum_{j=1}^{q-1}\langle -\frac{1}{2}J(Z_{n+1})X_q+\frac{1}{2}J(Z_q)X_{n+1},X_j\rangle\langle\frac{1}{2}[X_q,X_j],Z_k\rangle$$

$$=\frac{1}{4}\sum_{i=1}^{q}\sum_{j=1}^{q}\langle -J(Z_{n+1})X_i,X_j\rangle\langle[X_i,X_j],Z_k\rangle$$

$$=\frac{1}{4}\sum_{1\leqslant i\leqslant q,\, i=n+1}\;\sum_{1\leqslant j\leqslant q,\, j=n+1}\langle -J(Z_{n+1})X_i,X_j\rangle\langle J(Z_k)X_i,X_j\rangle$$

$$-\frac{1}{2}\sum_{1\leqslant i\leqslant q,\, i=n+1}\langle -J(Z_{n+1})X_{n+1},X_i\rangle\langle J(Z_k)X_{n+1},X_i\rangle$$

$$= \frac{1}{4} \sum_{1 \leqslant i \leqslant q,\, i = n+1} \langle -J(Z_{n+1})X_i, J(Z_k)X_i\rangle - \frac{1}{2}\langle -J(Z_{n+1})X_{n+1}, J(Z_k)X_{n+1}\rangle$$

$$= \frac{1}{4} \sum_{1 \leqslant i \leqslant q,\, i = n+1} \langle J(Z_k)J(Z_{n+1})X_i, X_i\rangle - \frac{1}{2}\langle J(Z_k)J(Z_{n+1})X_{n+1}, X_{n+1}\rangle$$

$$= -\operatorname{Ric}(Z_k, Z_{n+1}) + 2\langle R(X_{n+1}, Z_k)Z_{n+1}, X_{n+1}\rangle.$$

This completes the proof of (17) and of the lemma, as $Y_q = X_q - Z_q$, and $Y_{n+1} = X_{n+1} + Z_{n+1}$. □

P r o o f o f L e m m a 3. Let on U

$$E_i = \sum_{j=1}^{n+1} c_{ij} Y_j, \tag{19}$$

where c_{ij}, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n+1$ are scalar functions on U. Note that $E_i(p) = Y_i$, so $c_{ij}(p) = \delta_{ij}$. Using (19), we get

$$\begin{aligned} \nabla_{E_i} Y_{n+1} &= \sum_{j=1}^{n+1} c_{ij} \nabla_{Y_j} Y_{n+1} = \tfrac{1}{2} \sum_{j=1}^{q} c_{ij} \left([X_j, X_{n+1}] - J(Z_{n+1})X_j \right) \\ &+ \tfrac{1}{2} c_{iq} J(Z_q) X_{n+1} - \tfrac{1}{2} \sum_{j=q+1}^{n} c_{ij} J(Z_j) X_{n+1} - c_{i\,n+1} J(Z_{n+1}) X_{n+1}. \end{aligned} \tag{20}$$

Also, for $1 \leqslant k \leqslant n$ at p we have

$$\nabla_{E_k} E_i = \sum_{j=1}^{n+1} \left(Y_k(c_{ij}) Y_j + c_{ij}(p) \nabla_{Y_k} Y_j \right) = \sum_{j=1}^{n+1} Y_k(c_{ij}) Y_j + \nabla_{Y_k} Y_i. \tag{21}$$

In particular, at p

$$b_{ki}(p) = \langle \nabla_{E_k} E_i, \eta\rangle(p) = Y_k(c_{i\,n+1}) + \langle \nabla_{Y_k} Y_i, Y_{n+1}\rangle. \tag{22}$$

Considering (21) for $k = i$, projecting both sides of it to T_pM, and using the properties of the geodesic frame, we get

$$0 = \sum_{j=1}^{n} Y_i(c_{ij}) Y_j + \left(\nabla_{Y_i} Y_i\right)^T.$$

For $1 \leqslant i \leqslant q-1$ and $q+1 \leqslant i \leqslant n$ $\nabla_{Y_i} Y_i = 0$, and $\nabla_{Y_q} Y_q = J(Z_q)X_q = \left(\nabla_{Y_q} Y_q\right)^T$, since $\langle J(Z_q)X_q, X_{n+1} + Z_{n+1}\rangle = 0$. Then, for $1 \leqslant j \leqslant q$ we obtain

$$Y_i(c_{ij}) = \left[\begin{array}{ll} 0, & 1 \leqslant i \leqslant q-1; \\ -\langle J(Z_q)X_q, Y_j\rangle, & i = q; \\ 0, & q+1 \leqslant i \leqslant n. \end{array}\right.$$

We can deduce from (22) and the above considerations that for $1 \leqslant i \leqslant n$ $b_{ii}(p) = Y_i(c_{i\,n+1})$. Differentiate (20) with respect to E_i at p. For $1 \leqslant i \leqslant q-1$ we get

$$\nabla_{E_i}\nabla_{E_i} Y_{n+1} = -Y_i(c_{i\,n+1})J(Z_{n+1})X_{n+1} + \frac{1}{2}\nabla_{X_i}\left([X_i, X_{n+1}] - J(Z_{n+1})X_i\right)$$

$$= -b_{ii}(p)J(Z_{n+1})X_{n+1} - \frac{1}{4}J([X_i, X_{n+1}])X_i - \frac{1}{4}[X_i, J(Z_{n+1})X_i].$$

For $i = q$ we have

$$\nabla_{E_q}\nabla_{E_q} Y_{n+1} = -Y_q(c_{i\,n+1})J(Z_{n+1})X_{n+1} + \sum_{j=1}^{n} Y_q(c_{qj})\nabla_{Y_j} Y_{n+1}$$

$$+\frac{1}{2}\nabla_{Y_q}\left([X_q, X_{n+1}] - J(Z_{n+1})X_q + J(Z_q)X_{n+1}\right) = -b_{qq}(p)J(Z_{n+1})X_{n+1}$$

$$-\frac{1}{2}\sum_{j=1}^{q-1}\langle J(Z_q)X_q, X_j\rangle\left([X_j, X_{n+1}] - J(Z_{n+1})X_j\right)$$

$$-\frac{1}{4}[X_q, J(Z_{n+1})X_q] + \frac{1}{4}[X_q, J(Z_q)X_{n+1}] - \frac{1}{4}J(Z_q)J(Z_{n+1})X_q + \frac{1}{4}J(Z_q)^2 X_{n+1}.$$

For $q+1 \leqslant i \leqslant n$ we obtain

$$\nabla_{E_i}\nabla_{E_i} Y_{n+1} = -Y_i(c_{i\,n+1})J(Z_{n+1})X_{n+1} - \frac{1}{2}\nabla_{Z_i}\left(J(Z_i)X_{n+1}\right)$$

$$= -b_{ii}(p)J(Z_{n+1})X_{n+1} + \frac{1}{4}J(Z_i)^2 X_{n+1}.$$

Summing up these expressions, we get for $1 \leqslant k \leqslant q-1$

$$-\sum_{i=1}^{n}\langle\nabla_{E_i}\nabla_{E_i} Y_{n+1}, X_k\rangle = nH(p)\langle J(Z_{n+1})X_{n+1}, X_k\rangle$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{i=1}^{q-1}\langle J([X_i, X_{n+1}])X_i, X_k\rangle + \frac{1}{2}\sum_{j=1}^{q-1}\langle J(Z_q)X_q, X_j\rangle\langle -J(Z_{n+1})X_j, X_k\rangle \\
&+\frac{1}{4}\langle J(Z_q)J(Z_{n+1})X_q, X_k\rangle - \frac{1}{4}\langle J(Z_q)^2X_{n+1}, X_k\rangle - \frac{1}{4}\sum_{i=q+1}^{n}\langle J(Z_i)^2X_{n+1}, X_k\rangle \\
&= nH(p)\langle J(Z_{n+1})X_{n+1}, X_k\rangle - \frac{1}{2}\sum_{1\leqslant i\leqslant q,\, i=n+1}\langle J([X_{n+1}, X_i])X_i, X_k\rangle \\
&= nH(p)\langle J(Z_{n+1})X_{n+1}, X_k\rangle - \mathrm{Ric}(X_k, X_{n+1}).
\end{aligned}
$$

Here we use the equation $J(Z_q)J(Z_{n+1})X_q = J(Z_{n+1})^2X_{n+1}$, which follows from the construction of the frame. Thus we obtain the first expression in (18).

For $q+1 \leqslant k \leqslant n$ we have

$$
\begin{aligned}
-\sum_{i=1}^{n}\langle \nabla_{E_i}\nabla_{E_i}Y_{n+1}, Z_k\rangle &= \frac{1}{4}\sum_{i=1}^{q-1}\langle [X_i, J(Z_{n+1})X_i], Z_k\rangle \\
&+\frac{1}{2}\sum_{j=1}^{q-1}\langle J(Z_q)X_q, X_j\rangle\langle [X_j, X_{n+1}], Z_k\rangle \\
&+\frac{1}{4}\langle [X_q, J(Z_{n+1})X_q], Z_k\rangle - \frac{1}{4}\langle [X_q, J(Z_q)X_{n+1}], Z_k\rangle \\
= -\frac{1}{4}\sum_{1\leqslant i\leqslant q,\, i=n+1}&\langle J(Z_k)J(Z_{n+1})X_i, X_i\rangle + \langle J(Z_k)J(Z_{n+1})X_{n+1}, X_{n+1}\rangle \\
&= \mathrm{Ric}(Z_k, Z_{n+1}) - 4\langle R(X_{n+1}, Z_k)Z_{n+1}, X_{n+1}\rangle.
\end{aligned}
$$

In the above calculation we used the fact that $J(Z_q)X_q = J(Z_{n+1})X_{n+1}$ and $[X_q, J(Z_q)X_{n+1}] = [X_{n+1}, J(Z_{n+1})X_{n+1}]$. As $Y_q = X_q - Z_q$ and $Y_{n+1} = X_{n+1} + Z_{n+1}$, we get the last three equalities in (18). □

3 Mean curvature and harmonicity

Consider the tangent bundle TN and the distribution in TN formed by left invariant vector fields from $\mathcal{Z}$. Since $\mathcal{Z}$ is an abelian ideal, we can integrate this distribution and obtain a foliation. Denote this foliation by $\mathcal{F}_{\mathcal{Z}}$. Let G be harmonic. Since by (8), in this case $Y_k(nH) = 0$ for all $q+1 \leqslant k \leqslant n$, we have

Corollary 4. *If the Gauss map of M is harmonic, then for each leaf M' of $\mathcal{F}_{\mathcal{Z}}$ the mean curvature of the immersion is constant on $M \cap M'$.*

Now we obtain some analogues of the results for Lie groups with bi-invariant metrics that were stated in [6].

Let ν be a vector field on M defined by $\nu(p) = Y_q$, for $p \in M$. In other words, we obtain $\nu(p)$ rotating the unit normal vector $\eta(p)$ by the angle $\frac{\pi}{2}$ in the 2-plane containing $\eta(p)$ and orthogonal to both $dL_p(\mathcal{V})$ and $dL_p(\mathcal{Z})$.

Proposition 5. *Let M be a compact smooth oriented hypersurface in a 2-step nilpotent Lie group N. Assume that*

(i). the mean curvature of M is constant on the integral curves of ν;

(ii). the Gauss map of M is harmonic;

(iii). $\|B\|^2 + \mathrm{Ric}(\eta, \eta) \geqslant 0$ on M and $\|B\|^2 + \mathrm{Ric}(\eta, \eta) > 0$ in some point of M;

(iv). the set of points $p \in M$ such that $\eta(p) \notin dL_p(\mathcal{V})$ is dense in M.

Then $G(M)$ is contained in a closed hemisphere of S^n if and only if $G(M)$ is contained in a great sphere of S^n.

P r o o f. One of the implications in the proposition is obvious. Suppose that some closed hemisphere of S^n contains $G(M)$, i.e., there exists a unit vector $v \in \mathbb{R}^{n+1}$ such that for all $p \in M$ $\langle G(p), v\rangle$ is nonpositive. Consider a smooth function $f = \langle G, v\rangle$ on M. The coefficient of $Y_q(e)$ in (8) vanishes. For all points from some dense set of M we have $X_q \neq 0$ and thus $X_{n+1} = \frac{|X_{n+1}|}{|X_q|} X_q$. This, together with $Y_q(nH) = 0$, implies that the coefficient of $Y_{n+1}(e)$ is equal to $-\|B\|^2 - \mathrm{Ric}(\eta, \eta)$ on the dense subset of M and hence on the whole M because both the coefficient and $-\|B\|^2 - \mathrm{Ric}(\eta, \eta)$ are continuous. Taking the scalar product of (8) with v, we obtain

$$\Delta f = -\Big(\|B\|^2 + \mathrm{Ric}(\eta, \eta)\Big) f \geqslant 0.$$

Then f is a subharmonic function on the compact manifold M. Thus f is constant, and $\Big(\|B\|^2 + \mathrm{Ric}(\eta, \eta)\Big) f = -\Delta f = 0$. From the hypothesis, this implies $f = 0$, hence $G(M)$ is contained in the equator $v^{\perp}$. This completes the proof. $\square$

Proposition 6. *Suppose that a smooth oriented hypersurface M in a 2-step nilpotent Lie group N is CMC, its Gauss map is harmonic, for all p from some dense set of M the normal vector $\eta(p) \notin dL_p(\mathcal{V})$, and $G(M)$ is contained in an open hemisphere of S^n. Then M is stable.*

P r o o f. From the hypothesis, there exists $v \in \mathbb{R}^{n+1}$ such that for all $p \in M$ $\langle G(p), v\rangle > 0$. As in the proof of Proposition 5, consider a scalar function $w(p) = \langle G(p), v\rangle$ on M. This function is smooth and positive. As above, (8) implies the Jacobi equation $(\Delta + \|B\|^2 + \mathrm{Ric}(\eta,\eta))\, w = 0$. Now [7, Theorem 1] implies the stability of M. $\square$

4 Groups of Heisenberg type

Let N be a group of Heisenberg type. Then from (5), for all $X, Y \in \mathcal{V}$,

$$\mathrm{Ric}(X,Y) = \frac{1}{2}\sum_{k=q}^{n+1}\langle J(Z_k)^2 X, Y\rangle = -\frac{1}{2}(n+1-q)\langle X, Y\rangle.$$

Also, we can rewrite the coefficients in (8) for $1 \leqslant k \leqslant q$ and for $k = n+1$ in the form

$$\sum_{j=1}^{q-1}\langle J([X_k, X_j])X_j, X_{n+1}\rangle + 4\langle R(X_k, Z_{n+1})Z_{n+1}, X_{n+1}\rangle$$

$$= \left[\begin{array}{ll} 0, & 1 \leqslant k \leqslant q-1; \\ |Z_{n+1}|\,|X_{n+1}|\left(q-n-1+|Z_{n+1}|^2\right), & k = q; \\ |X_{n+1}|^2\left(q-n-1+|Z_{n+1}|^2\right), & k = n+1. \end{array}\right.$$

Moreover,

$$\mathrm{Ric}(Z_{n+1}, Z_{n+1}) = -\frac{1}{4}\,\mathrm{Tr}\, J(Z_{n+1})^2 = \frac{q}{4}\,|Z_{n+1}|^2\,,$$

and thus

$$\mathrm{Ric}(Y_{n+1}, Y_{n+1}) = \frac{q}{4}\,|Z_{n+1}|^2 - \frac{1}{2}(n+1-q)\,|X_{n+1}|^2\,.$$

Equation (8) now takes the form

$$
\begin{aligned}
\Delta G(p) = \sum_{k=1}^{q-1} \Bigg(& -Y_k(nH) - 2\sum_{i=1}^{q}\sum_{j=q+1}^{n} b_{ij}(p)\langle J(Z_j)X_i, X_k\rangle \\
& + 2\sum_{i=1}^{q} b_{iq}(p)\langle J(Z_q)X_i, X_k\rangle + nH(p)\langle J(Z_{n+1})X_{n+1}, X_k\rangle \Bigg) Y_k(e) \\
& + \Bigg(- Y_q(nH) + |Z_{n+1}|\,|X_{n+1}|\left(q-n-1+|Z_{n+1}|^2\right) \\
& - 2\sum_{i=1}^{q}\sum_{j=q+1}^{n} b_{ij}(p)\langle J(Z_j)X_i, X_q\rangle \\
& +2\sum_{i=1}^{q} b_{iq}(p)\langle J(Z_q)X_i, X_q\rangle + nH(p)\langle J(Z_{n+1})X_{n+1}, X_q\rangle \Bigg) Y_q(e) \\
& + \sum_{k=q+1}^{n} \Bigg(- Y_k(nH) \Bigg) Y_k(e) \\
& + \Bigg(-2\sum_{i=1}^{q}\sum_{j=q+1}^{n} b_{ij}(p)\langle J(Z_j)X_i, X_{n+1}\rangle \\
& + 2\sum_{i=1}^{q} b_{iq}(p)\langle J(Z_q)X_i, X_{n+1}\rangle - \|B\|^2(p) - \tfrac{q}{4}\,|Z_{n+1}|^2 \\
& + |X_{n+1}|^2\left(\tfrac{1}{2}(q-n-1) + |Z_{n+1}|^2\right) \Bigg) Y_{n+1}(e).
\end{aligned}
\tag{23}
$$

Consider the case $n = q$, i.e., $\dim \mathcal{Z} = 1$. It is easy to see that n is then even, $n = 2m$, where m is a positive integer, and N is isomorphic to the $2m+1$-dimensional Heisenberg group (recall that N is connected and simply connected).

In this case at p we can choose $X_1, \ldots, X_{2m+1}$ so that

$$
\begin{gathered}
J(Z)X_i = X_{m+i},\ 1 \leqslant i \leqslant m-1; \\
J(Z)X_m = \frac{X_{2m}}{|X_{2m}|} = \frac{X_{2m}}{|Z_{2m+1}|} \text{ if } Z_{2m+1} \neq 0; \text{or} \frac{X_{2m+1}}{|X_{2m+1}|} \text{ if } X_{2m+1} \neq 0; \\
J(Z)X_{m+i} = -X_i,\ 1 \leqslant i \leqslant m-1; \\
J(Z)X_{2m} = -|X_{2m}|\,X_m = -|Z_{2m+1}|\,X_m; \\
J(Z)X_{2m+1} = -|X_{2m+1}|\,X_m.
\end{gathered}
$$

Choose $Z_{2m} = |X_{2m+1}|\, Z$ and $Z_{2m+1} = |Z_{2m+1}|\, Z$. Then (23) has the form

$$
\begin{aligned}
\Delta G(p) = -\sum_{k=1}^{m-1} \Big(& Y_k(2mH) + 2b_{2m\,m+k}(p)\, |X_{2m+1}| \Big) Y_k(e) \\
& - \Big(Y_m(2mH) + 2mH(p)\, |X_{2m+1}| \\
& + 2b_{2m\,2m}(p)\, |X_{2m+1}|\, |Z_{2m+1}| \Big) Y_m(e) \\
& - \sum_{k=1}^{m-1} \Big(Y_{m+k}(2mH) - 2b_{2m\,k}(p)\, |X_{2m+1}| \Big) Y_k(e) \\
& - \Big(Y_{2m}(2mH) + |X_{2m+1}|^3\, |Z_{2m+1}| \\
& - 2b_{2m\,m}(p)\, |X_{2m+1}|\, |Z_{2m+1}| \Big) Y_{2m}(e) \\
& - \Big(\|B\|^2(p) + \tfrac{m}{2}\, |Z_{2m+1}|^2 - \tfrac{1}{2}\, |X_{2m+1}|^2 \\
& + |X_{2m+1}|^4 - 2b_{2m\,m}(p)\, |X_{2m+1}|^2 \Big) Y_{2m+1}(e).
\end{aligned}
\tag{24}
$$

Consider an example of the three-dimensional Heisenberg group Nil. In the space $\mathbb{R}^3$ with Cartesian coordinates (x, y, z), define vector fields

$$X = \frac{\partial}{\partial x},\ Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ Z = \frac{\partial}{\partial z}\,.$$

Then $\mathrm{Span}(X, Y, Z)$ is a Lie algebra (with the only nonzero bracket $[X, Y] = Z$), which is the Lie algebra of Nil. Introduce a scalar product in such a way that the vectors X, Y and Z are orthonormal. Consider the following unit vector field:

$$\eta = \frac{xY + Z}{\sqrt{1+x^2}},$$

and vector fields

$$F_1 = X,\ F_2 = \frac{Y - xZ}{\sqrt{1+x^2}},$$

which are orthogonal to η. In the notation of section 2, in each p $F_1 = X_1$, $F_2 = X_2 - Z_2$, $\eta = X_3 + Z_3$. By direct computation of covariant derivatives it can be shown that the distribution spanned by F_1 and F_2 is integrable and form the tangent bundle of some two-dimensional foliation $\mathcal{F}$ in Nil. From the computation of the second fundamental form we obtain $\|B\|^2 = \frac{\left(x^2-1\right)^2}{2(1+x^2)^2}$,

and $H = 0$. Thus the leaves of this foliation are minimal surfaces. The Laplacian on $G = \eta$ is

$$\Delta G = \left(F_1 F_1 + F_2 F_2 - (\nabla_{F_1} F_1)^T - (\nabla_{F_2} F_2)^T \right) G$$

$$= -\frac{x}{(1+x^2)^2} F_2 - \frac{1}{(1+x^2)^2} \eta.$$

We obtain the same result considering (24) at some p. In fact,

$$2b_{22} |X_3| |Z_3| = 0;$$

$$|X_3|^3 |Z_3| - 2b_{21} |X_3| |Z_3| = \frac{x}{(1+x^2)^2};$$

$$\|B\|^2 + \frac{1}{2} |Z_3|^2 - \frac{1}{2} |X_3|^2 + |X_3|^4 - 2b_{21} |X_3|^2 = \frac{1}{(1+x^2)^2}.$$

In particular, foliation $\mathcal{F}$ gives an example of a CMC-surface in Nil such that its Gauss map is not harmonic.

Proposition 7. *Suppose that M is a smooth oriented $2m$-dimensional manifold immersed in the $2m+1$-dimensional Heisenberg group. If any two of the following three claims are true, then the third one is also true.*

(i). M is CMC;

(ii). the Gauss map of M is harmonic;

(iii). at every point of M, the following holds:

$$\begin{cases} b_{2m\,k} = 0, 1 \leqslant k \leqslant m-1,\ m+1 \leqslant k \leqslant 2m-1; \\ |Z_{2m+1}| \left(|X_{2m+1}|^2 - 2b_{2m\,m} \right) = 0; \\ |Z_{2m+1}| \left(b_{1\,1} + \cdots + b_{2m-1\,2m-1} + 3b_{2m\,2m} \right) = 0. \end{cases} \tag{25}$$

Here b_{ij}, $1 \leqslant i, j \leqslant 2m$ are the coefficients of the second fundamental form of M in the basis chosen as above.

P r o o f. If *(iii)* is true, then the equivalency of *(i)* and *(ii)* immediately follows from 24. Suppose *(i)* and *(ii)* are true. Let A be a set of such points of M that $|X_{2m+1}| \neq 0$. At the points of A 24 implies the expressions in (25) . Since the distribution orthogonal to Z is non-integrable, A is dense in M. Now the continuity of the left hand sides of the equations (25) implies *(iii)*. □

In the case $m = 1$ the next theorem shows that the restrictions for M arising from (25) are rather strict.

Theorem 8. *Let M be a smooth oriented CMC-surface in the Heisenberg group Nil whose Gauss map is harmonic. Then M is a "cylinder", that is, its position vector in the coordinates x, y, z has the form*

$$r(s,t) = (f_1(s), f_2(s), t), \tag{26}$$

where f_1 and f_2 are some smooth functions.

P r o o f. For each $p \in M$ denote $a(p) = |X_3|$, $b(p) = |Z_3|$. Then a and b are smooth scalar functions on M, and $a^2 + b^2 = 1$. Consider an arbitrary point p of M. Choose X_1 as above and put $X_2 = J(Z)X_1$. Denote by T_1 and T_2 the vector fields that at each $p \in M$ are equal to X_1 and X_2 respectively. Consider unit tangent vector fields F_1 and F_2, and a unit normal vector field η of M of the form

$$F_1 = T_1,\ F_2 = bT_2 - aZ,\ \eta = aT_2 + bZ.$$

Denote by κ_1 and κ_2 the geodesic curvatures of the integral curves of F_1 and F_2 respectively. In other words,

$$\overline{\nabla}_{F_1}F_1 = \kappa_1 F_2,\ \overline{\nabla}_{F_1}F_2 = -\kappa_1 F_1,\ \overline{\nabla}_{F_2}F_1 = -\kappa_2 F_2,\ \overline{\nabla}_{F_2}F_2 = \kappa_2 F_1, \tag{27}$$

where $\overline{\nabla}$ is the Riemannian connection on M induced by the immersion. The Gaussian curvature of the surface is

$$K = F_1(\kappa_2) + F_2(\kappa_1) - (\kappa_1)^2 - (\kappa_2)^2\,. \tag{28}$$

Assume that for some $p \in M$ $a(p) \neq 0$ and $b(p) \neq 0$. Then $ab \neq 0$ on some neighborhood U of p. Then (25) implies that on U the matrix of the second fundamental form of M is

$$\begin{pmatrix} 3H & \frac{1}{2}a^2 \\ \frac{1}{2}a^2 & -H \end{pmatrix}. \tag{29}$$

In particular, the extrinsic curvature K_{ext} of the surface is $-3H^2 - \frac{1}{4}a^4$.

Denote by B the second fundamental form of the immersion. Then the Codazzi equations for M are

$$(\nabla_{F_1}B)(F_2,F_1) - (\nabla_{F_2}B)(F_1,F_1) = \langle R(F_1,F_2)F_1,\eta\rangle = ab;$$

$$(\nabla_{F_2}B)(F_1,F_2) - (\nabla_{F_1}B)(F_2,F_2) = \langle R(F_2,F_1)F_2,\eta\rangle = 0.$$

Computing the covariant derivatives of the second fundamental form, we obtain for U

$$\begin{aligned} aF_1(a) + 4H\kappa_1 - a^2\kappa_2 - ab = 0,\\ aF_2(a) - 4H\kappa_2 - a^2\kappa_1 = 0. \end{aligned} \tag{30}$$

The Gauss equation has the form

$$K = K_{ext} + \langle R(F_1,F_2)F_2,F_1\rangle = -3H^2 - \frac{1}{4}a^4 - \frac{3}{4}b^2 + \frac{1}{4}a^2.$$

From (28) we obtain

$$F_1(\kappa_2) + F_2(\kappa_1) - (\kappa_1)^2 - (\kappa_2)^2 = -3H^2 - \frac{1}{4}a^4 - \frac{3}{4}b^2 + \frac{1}{4}a^2. \tag{31}$$

Using (30) and the form of F_2 and η, we can derive

$$\langle \nabla_{F_1}F_2,\eta\rangle = -\langle F_2,\nabla_{F_1}\eta\rangle = -\langle F_2,\nabla_{F_1}\left(\frac{a}{b}F_2 + \left(\frac{a^2}{b}+b\right)Z\right)\rangle$$

$$= -F_1\left(\frac{a}{b}\right)\langle F_2,F_2\rangle - F_1\left(\frac{1}{b}\right)\langle F_2,Z\rangle - \frac{a}{b}\langle F_2,\nabla_{F_1}F_2\rangle - \frac{1}{b}\langle F_2,\nabla_{F_1}Z\rangle$$

$$= -F_1\left(\frac{a}{b}\right) + aF_1\left(\frac{1}{b}\right) - \frac{1}{b}\langle F_2,-\frac{1}{2}T_2\rangle = -\frac{1}{b}\left(-\frac{4H\kappa_1}{a} + a\kappa_2 + b\right) + \frac{1}{2}$$

$$= -\frac{1}{2} + \frac{4H\kappa_1}{ab} - \frac{a}{b}\kappa_2;$$

$$\langle \nabla_{F_2}F_1,\eta\rangle = -\langle F_1,\nabla_{F_2}\eta\rangle = -\langle F_1,\nabla_{F_2}\left(\frac{a}{b}F_2 + \left(\frac{a^2}{b}+b\right)Z\right)\rangle$$

$$= -\frac{a}{b}\langle F_1,\nabla_{F_2}F_2\rangle - \frac{1}{b}\langle F_1,\nabla_{F_2}Z\rangle = -\frac{a}{b}\kappa_2 - \frac{1}{b}\langle T_1,\frac{1}{2}bT_1\rangle = -\frac{a}{b}\kappa_2 - \frac{1}{2}.$$

In the above equations we used the fact that Z is left invariant and the expressions (2) for the covariant derivative. Since $ab \neq 0$, the integrability condition $\langle [F_1, F_2], \eta \rangle = 0$ takes the form $H\kappa_1 = 0$. Besides, (29) imply

$$3H = b_{11} = \langle \nabla_{F_1} F_1, \eta \rangle = -\langle F_1, \nabla_{F_1} \eta \rangle$$

$$= -\langle F_1, \nabla_{F_1} \left(\frac{a}{b} F_2 + \left(\frac{a^2}{b} + b \right) Z \right) \rangle = -\frac{a}{b} \langle F_1, \nabla_{F_1} F_2 \rangle - \frac{1}{b} \langle F_1, \nabla_{F_1} Z \rangle = \frac{a}{b} \kappa_1.$$

Thus $H = \kappa_1 = 0$. In particular, $\nabla_{F_1} F_1 = 0$, hence $T_1 = F_1$ is a geodesic vector field in the ambient manifold. Note that T_1 belongs to the distribution that spans the left invariant vector fields of $\mathcal{V}$. Considering the set of geodesics in Nil (see [3, proposition (3.1), proposition (3.5)]), we obtain that $T_1 = cX + dY$, where $c, d \in \mathbb{R}$ are some constants, i.e., $T_1 = X_1$ and $T_2 = X_2$ are left invariant. Note that the second equation of (30) implies $F_2(a) = F_2(b) = 0$. Thus we obtain

$$\nabla_{F_2} F_2 = \nabla_{F_2} (bX_2 - aZ) = b \nabla_{bX_2 - aZ} X_2 - a \nabla_{bX_2 - aZ} Z = -abX_1.$$

Therefore $\kappa_2 = -ab$. It follows from this equation, from the computations above in this proof, and from (29) that

$$\frac{1}{2} a^2 = b_{12} = \langle \nabla_{F_1} F_2, \eta \rangle = -\frac{a}{b} \kappa_2 - \frac{1}{2} = a^2 - \frac{1}{2},$$

and $a^2 = b^2 = \frac{1}{2}$. But then $a = b = \frac{\sqrt{2}}{2}$, and the first equation in (30) implies $a\kappa_2 + b = 0$, which leads to a contradiction.

Thus $ab = 0$ at each point of M. Since $a^2 + b^2 = 1$ and a, b are continuous, $a = 1$ or $b = 1$ identically. The latter case is impossible because $Z^\perp$ is not integrable; then the normal vector of M is orthogonal to $\mathcal{Z}$, and $F_2 = -Z$. Therefore M is invariant under the action of $\mathcal{Z}$ by left translations, and M is formed by integral curves of Z, which are geodesics $(0, 0, t)$. Then M has the form (26). $\square$

Note that similar result for another definition of the Gauss map was obtained in [13]. Also, in [13] the equations of the CMC-surfaces of the form (26) were obtained. Proposition 7 then implies that the Gauss maps of all these surfaces are harmonic.

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